Who's Who in Networks. WANTED: The Key Player

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May 31, 2022

Motivation

- A network consists of several individuals linking to each other or not, and there may be some groups in a network.
- The dependence of individual outcomes on group behavior is often referred to as peer effects.
  - In standard peer effects models, this dependence is homogeneous across memebrs and corresponds to an average group influence.
  - As a decision-maker or policymaker, we may want to find the most influential player in the network to break or strengthen such effect.
- What if this intergroup externality is heterogeneous cross group members and varies accross individuals with their level of group exposure?

Literature Reviews

- The first related measure was proposed by Bonacich (1987), and some sociologists establish the network analysis Wasserman and Faust (1994) as well.
- However, the Bonacich centrality measure fails to internalize all the network payoff externalities agents exert on each other, whereas the intercentrality measure internalizes them all.
- This research extended the Bonacich centrality measure and propose a new centrality measure based on the planner's optimality (collective) perspectives instead of strategic (individual) considerations.

Outline

1. Model Setting
2. Equilibrium Analysis
3. Find the Key Player
4. Discussion

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Utility and the Game

- Each player $i = 1, \cdots, n$ selects an effort $x_i \geq 0$ and obtains the bilinear utility $u_i(x_1, \cdots, x_n) = \alpha_i x_i + \frac{1}{2}\sigma_{ii} x_i^2 + \sum_{j \neq i} \sigma_{ij} x_i x_j$, which is strictly concave in own effort, and the utility is linear-quadratic.
- Bilateral influences are captured by the cross-derivatives $\frac{\partial^2 u_i}{\partial x_i \partial x_j} = \sigma_{ij}$ and can be of either sign.
  - For example, if $\sigma_{ij} > 0$, an increase in j's efforts triggers a upwards shift in i's response, and we say i and j's efforts are strategic complements from i's perspective.
- Simplifying, we set $\alpha_i = \alpha > 0$, $\sigma_{ii} = \sigma$, and denote by $\mathbf{\Sigma} \equiv [\sigma_{ij}]$ the square matrix of cross-effects.
- Moreover, we define $\underline{\sigma} \equiv \min\{\sigma_{ij} | i \neq j\}$ and $\bar{\sigma} \equiv \max\{\sigma_{ij} | i \neq j\}$ and assume that $\sigma < \min\{\underline{\sigma}, 0\}$.

Cross-effects

- The next step is to discuss how to capture the relative complementarity in efforts between (i, j).
  - There are some discussion based on the sign of $\underline{\sigma}$, and we skip it and use the result directly.
- Define $\gamma \equiv -\min\{\underline{\sigma}, 0\} \geq 0$ and $\lambda \equiv \bar{\sigma} + \gamma \geq 0$. [1] and let $g_{ij} \equiv \frac{\sigma_{ij}+\gamma}{\lambda}$ for $i \neq j$ and $g_{ii} = 0$. [2] Therefore, $0 \leq g_{ij} \leq 1$ is a parameter measuring the relationship in efforts within (i, j) from i's perspective, and the matrix $\boldsymbol{G} = [g_{ij}]$ interprets the adjacency matrix of the network.

[1] In fact, $\lambda = 0$ has Lebesgue measure zero.

[2] The result is robust in the case $g_{ii} = 1$. This case is less economic intuitive said by the author.

Bilateral Influences

- Let $\sigma = -\beta - \gamma$ for $\beta > 0$ satisfying the assumption of $\sigma < \min\{\underline{\sigma}, 0\}$ WLOG, and denote by $\boldsymbol{I}$ the identity matrix and $\boldsymbol{U}$ the matrix of ones, where both are $n \times n$ matrices, we can decompose the matrix $\boldsymbol{\Sigma}$ as $\boldsymbol{\Sigma} = -\beta\boldsymbol{I} - \gamma\boldsymbol{U} + \lambda\boldsymbol{G}$.
  - Therefore, bilateral influences result from the combination of an individual effect by $-\beta\boldsymbol{I}$, the global interaction effect by $-\gamma\boldsymbol{U}$, and the local interaction effect by $\lambda\boldsymbol{G}$.
- We can rewrite the utility function following the decomposition of $\boldsymbol{\Sigma}$ as $u_i(x_1, \cdots, x_n) = \alpha x_i - \frac{1}{2}(\beta - \gamma)x_i^2 - \gamma\sum_{j=1}^{n} x_i x_j + \lambda\sum_{j=1}^{n} g_{ij}x_i x_j$ for all $i = 1, \cdots, n$.

The Bonacich centrality measure

- Before moving to the equilibrium analysis, we define a network centrality measure extended by Bonacich centrality measure for the further use.
- Remind that the matrix $\boldsymbol{G}^k$ tracks the indirect connections in the network: g_{ij}^k measures the number of paths of length $k \geq 1$ in the network $\mathcal{G}$ from i to j.
- Given a sufficiently small scalar $a \geq 0$, we define the matrix $\boldsymbol{M}(\mathcal{G}, a) = [\boldsymbol{I} - a\boldsymbol{G}]^{-1} = \sum_{k=0}^{+\infty} a^k \boldsymbol{G}^k$. a represents a decay factor to scale down the weight of long paths.
- The vector of Bonacich centrality in $\mathcal{G}$ is $b(\mathcal{G}, a) = [\boldsymbol{I} - a\boldsymbol{G}]^{-1} \cdot \mathbb{1}$, and the Bonacich centrality of node i is $b_i(\mathcal{G}, a) = \sum_{j=1}^{n} m_{ij}(\mathcal{G}, a)$.

The Bonacich centrality measure

- We can separate the Bonacich centrality into two parts: from i to i itself and of all the outer path from i to every other $j \neq i$. That is, $b_i(\mathcal{G}, a) = \sum_{j=1}^{n} m_{ij}(\mathcal{G}, a) = m_{ii}(\mathcal{G}, a) + \sum_{j \neq i} m_{ij}(\mathcal{G}, a)$.
  - $m_{ii}(\mathcal{G}, a) \geq 1$ by definition and thus $b_i(\mathcal{G}, a) \geq 1$.

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Nash Equilibrium

- Recall that the utility function can be describe as $u_i(\boldsymbol{x}) = \alpha_i x_i + \frac{1}{2}\boldsymbol{\Sigma x}^2$. A Nash equilibrium in pure strategies $\boldsymbol{x}^* \in \mathbb{R}^n_+$ is to solve $\frac{\partial u_i(\boldsymbol{x}^*)}{\partial x_i} = 0$ and $x_i^* > 0$, that is, $-\boldsymbol{\Sigma} \cdot \boldsymbol{x}^* = [\beta \boldsymbol{I} + \gamma \boldsymbol{U} - \lambda \boldsymbol{G}] \cdot \boldsymbol{x}^* = \alpha \cdot \mathbb{1}$.
- Using the fact that $\boldsymbol{U} \cdot \boldsymbol{x}^* = x^* \cdot \mathbb{1}$ and define $\lambda^* \equiv \frac{\lambda}{\beta}$, the FOC reduces to $\beta[\boldsymbol{I} - \lambda^* \boldsymbol{G}] \cdot \boldsymbol{x}^* = (\alpha - \gamma x^*) \cdot \mathbb{1}$.

Theorem 1: Let $\mu_1(\boldsymbol{G})$ be the largest eigenvalue of $\boldsymbol{G}$, [3] the matrix $\beta[\boldsymbol{I} - \lambda^* \boldsymbol{G}]$ is well-defined and nonnegative if and only if $\beta > \lambda \mu_1(\boldsymbol{G})$, thus the unique interior Nash equilibrium is given by $\boldsymbol{x}^*(\boldsymbol{\Sigma}) = \frac{\alpha}{\beta + \gamma b(\mathcal{G}, \lambda^*)} \boldsymbol{b}(\mathcal{G}, \lambda^*)$.

[3] $\mu_1(\boldsymbol{G})$ is well-define and larger than 0 since all eigenvalues of a symmetric matrix $\boldsymbol{G}$ are real, and the diagnal of $\boldsymbol{G}$ is zero.

Parameters Analysis

- Given the unique Nash equilibrium $\boldsymbol{x}^*(\boldsymbol{\Sigma}) = \frac{\alpha}{\beta+\gamma b(\mathcal{G},\lambda^*)} \boldsymbol{b}(\mathcal{G}, \lambda^*)$, we want to analyze how three different effects influence the equilibrium.
  - If the matrix of cross-effects $\boldsymbol{\Sigma}$ reduces to $\lambda \boldsymbol{G}$, that is, $\beta = \gamma = 0$, there exists no Nash equilibrium.
  - If $\boldsymbol{\Sigma}$ reduces to $-\beta \boldsymbol{I} - \gamma \boldsymbol{U}$, that is, $\lambda = 0$, the Nash equilibrium is unique.
- The existence and uniqueness of Nash equilibrium are proven by Debreu and Herstein (1953). We emphasize the economic meaning.

  My explanation: If the cross-effects will not be affected by your effort and the substitutability in efforts across all pairs of players , you may prefer doing nothing and result in an effort $x_i = 0$ to obtain a higher utility, which contradicts the condition of an interior Nash equilibrium .

Individual's Contribution to the Aggregate Equilibrium

- The Bonacich-Nash equilibrium expression also implies that each individual contributes to the aggregate equilibrium outcome in proportion to their network centrality: $x_i^*(\mathbf{\Sigma}) = \frac{b_i(\mathcal{G},\lambda^*)}{b(\mathcal{G},\lambda^*)} x^*(\mathbf{\Sigma})$.
- This indicates that the intergroup externality is not an average influence but a weighted one heterogeneous across members.

  My explanation: An unbalanced influence across memebrs allows us to find the most significant player.

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Identification Criterion

- After solving the Nash equilibrium and related issues, we go back to the main topic: how to find the key player in a network.
- The idea is: we want to reduce the player optimally to maximize the difference between the value of aggregate Nash equilibrium from this removal. Formally, we solve an optimization problem $\max\{\boldsymbol{x}^*(\boldsymbol{\Sigma}) - \boldsymbol{x}^*(\boldsymbol{\Sigma}_{-i})\}$.
  - This is equivalent to solve $\min\{\boldsymbol{x}^*(\boldsymbol{\Sigma}_{-i}) | i = 1, \cdots, n\}$.
- Let i^* be a solution to the optimization problem. We call i^* the key player, which means removing i^* from the initial network has the largest overall impact on the aggregate equilibrium level.

New Measure: Intercentrality

- Remind that the Bonacich centrality measure only counts the number of paths stemming from player i, which doesn't include the contributions of player i toward other player $j \neq i$.
- Therefore, the author proposed the intercentrality $c_i(\mathcal{G}, a) = \frac{b_i(\mathcal{G},a)^2}{m_{ii}(\mathcal{G},a)}$, to capture such combined centrality.

$$
\begin{aligned}
\triangleright \quad c_i(\mathcal{G}, a) &= \frac{b_i(\mathcal{G}, a)^2}{m_{ii}(\mathcal{G}, a)} = \frac{\left(\sum_{j=1}^{n} m_{ij}(\mathcal{G}, a)\right)^2}{m_{ii}(\mathcal{G}, a)} \\
&= \frac{\left(m_{ii}(\mathcal{G}, a) + \sum_{j \neq i} m_{ij}(\mathcal{G}, a)\right)^2}{m_{ii}(\mathcal{G}, a)} \\
&= b_i(\mathcal{G}, a) + \frac{\sum_{j \neq i} m_{ij}(\mathcal{G}, a) \cdot b_i(\mathcal{G}, a)}{m_{ii}(\mathcal{G}, a)}.
\end{aligned}
$$

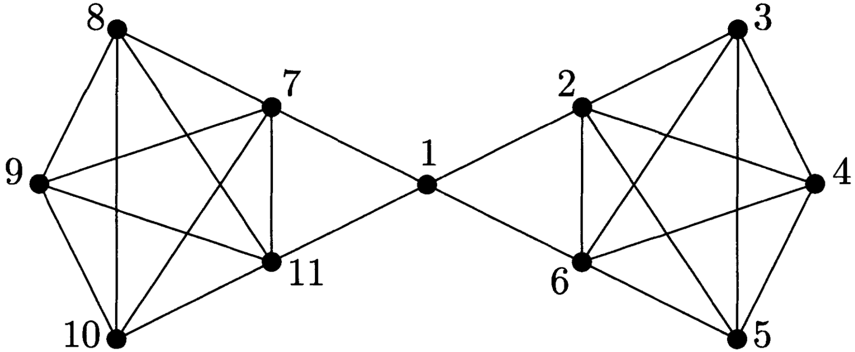
Intercentrality and the Key Player

- In fact, removing a player from a network has two effects:
  - Fewer players contribute to the aggregate activity level (direct effect).
  - The network topology is modified, which forces the remaining players to adopt different actions (indirect effect).
- Therefore, we want to catch the key play by using the intercentrality.

Theorem 2: The key player i^* who solves the optimization problem $\min\{x^*(\mathbf{\Sigma}_{-i})|i = 1, \cdots, n\}$ has the highest intercentrality of parameter λ^* in $\mathcal{G}$, that is, $c_{i^*}(\mathcal{G}, \lambda^*) \geq c_{-i^*}(\mathcal{G}, \lambda^*)$.

Example

- For example, consider the following network $\mathcal{G}$. Player 1 bridges together two groups, and removing player 1 disrupts the network.
- However, removing player 2 decreases maximally the total number of network links.



Example

- The computational result shows that as the value of a (the decay factor of long paths) is low, player 2 has the highest Bonacich centrality and also is the key player; however, when a is high, player 2 is not the key player but player 1 is.
- By considering indirect effects, removing player 1 has the highest joint direct and indirect effect on aggregate outcome.

	$a = 0.1$		$a = 0.2$	
Player Type	b_i	c_i	b_i	c_i
1	1.75	2.92	8.33	41.67*
2	1.88*	3.28*	9.17*	40.33
3	1.72	2.79	7.78	32.67

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Utility Form

- There is a number of possible extension of the work.
- The first is that the analysis is restricted to linear-quadratic utility that capture linear externality in player's actions.
  - They use FOC to find the interior equilibrium and leads to the Bonacich-Nash linkage.
- Linear-quadratic utilities are commonly used in economic models.
- It can be extended to more general cases, such as non-linear externalities.

Planner's Objective

- In this research, the planner's objective function is the aggregate group outcome. Theorems and corollaries are based on it.
- If the planer's objective is to maximize welfare $W^*(\mathbf{\Sigma}) = \sum_{i=1}^{n} u_i(\boldsymbol{x}^*(\mathbf{\Sigma})) = \frac{\beta+\gamma}{2\sum_{i=1}^{n} x_i^*(\mathbf{\Sigma})^2}$, the result of the key player is also possible in this case.

Group Targets

- This research characterizes a single-player target, but the idea of intercentrality measure can be generalized to a group index.
- The group target selection problem is not amenable to a sequential key player problem. In fact, optimal group targets belong to the maximization of submodular set functions, which cannot admit exact solutions.

Staged Games

- This method can be extended to solve a two-stage game.
    - In the first stage, players decide simultaneously to stay in the network $\mathcal{G}$ or to drop out of it, then get their outside options and utilities.
    - In the second stage, the staying players play the network game on the resulting network.
    - A fun fact is that the authors themselves had solved the uniqueness of the second-stage Nash equilibrium and the closed-form expression in Calvó-Armengoi and Zenou (2004) and Calvó-Armengol and Jackson (2004).

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