Assignment 1

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Problem 1

Let $T(i)$ be the sum of the i-th row of the Pascal triangle, we have $T(i) = 2^{i-1}$. The proof by induction is shown as below.

In the base case, when $i = 1$, the sum of the first row is 1, which is equal to 2^{1-1}. Assume the sum of row n is 2^{n-1}, then the element of row $n+1$ are each formed by adding two elements of row n, and each element of row n contributes to forming two elements of row $n+1$. Thus, the sum of the $n+1$ row is $2 \cdot 2^{n-1} = 2^n$ as acquired. By induction, we find the expression for the sum of the i-th row of the Pascal triangle.

Problem 2

In the base case, when $n = 0$, $H(2^0) = H(1) = 1 \geq 1$. Assume that $H(2^n) \geq 1 + \frac{2}{n}$ is true, when the case $n+1$, we have

$$\begin{aligned} H(2^{n+1}) &= 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} + \frac{1}{2^n+1} + \cdots + \frac{1}{2^{n+1}} \\ &= H(2^n) + \frac{1}{2^n+1} + \cdots + \frac{1}{2^{n+1}} \\ &\geq (1 + \frac{n}{2}) + \frac{1}{2^n+1} + \cdots + \frac{1}{2^{n+1}} \\ &\geq (1 + \frac{n}{2}) + 2^n \cdot \frac{1}{2^{n+1}} \\ &\geq (1 + \frac{n}{2}) + \frac{1}{2} \\ &\geq 1 + \frac{n+1}{2}. \end{aligned}$$

Since the case $n+1$ still satisfies the inequality, therefore, we prove the Harmonic series inequality by induction.

Problem 3

Let $P(n)$ be the proposition to carry out the proof. $P(n)$ is true if positive integer n can be written as a sum of distinct numbers from this series and be false if not.

When $1 \leq n \leq 10$, $P(n)$ is true evidently. Assume $n = k > 10$ can be written as a sum of distinct numbers from this series, when $n = k+1$, let a_i be the largest number in series and be less than $k+1$ simultaneously, that means $a_i < k+1$. Also, notice that $a_i > k+1-a_i$ since if $a_i \le k+1-a_i$, it implies $2a_i = a_{i+1} \le k+1$ *i.e.* a_{i+1} the largest number in series and be less than $k+1$ instead of a_i, contradicting to the assumption. Besides, $k > k+1-a_i$ implies $P(k+1-a_i)$ is true, which means $k+1-a_i$ can be written as a sum of distinct numbers from this series $a_{x_1} + a_{x_2} + ... + a_{x_j}$, and $a_i \notin \{a_{x_b} | b = 1, 2, \cdots j\}$. Therefore, $P(k+1)$ is true.

By induction, we prove that any positive integer can be written as a sum of distinct numbers from this series.

Problem 4

When $n = 1$, $F(1) = 1$ and $G(1) = 1 + 1 = 2$, the assumption of $G(n) = F(n) - 1$ is trivially incorrect since $G(1) = F(1) - 1$. However, the question statement lacks of definition of $G(1)$ and $G(2)$. Suppose we define $G(1) = G(2) = 0$ to imply $G(3) = 1$ and so on, the assumption is still dissatisfied.

Therefore, the critical problem of this proof is the incorrect assumption.

Problem 5

5.(a)

Given $Max(\perp) = 0$, the function Max is defined as below

$$Max(node(k, t_l, t_r)) = \begin{cases} k & \text{if } k \ge Max(t_l), Max(t_r) \\ Max(t_l) & \text{if } Max(t_l) > k, Max(t_r) \\ Max(t_r) & \text{if } Max(t_r) > k, Max(t_l). \end{cases}$$

5.(b)

After requiring $Max(\perp) = -1$, the definition of Max is modified as

$$Max(node(k, t_l, t_r)) = \begin{cases} -1 & \text{if } node(k, t_l, t_r) = \perp \\ k & \text{if } k \ge Max(t_l), Max(t_r) \\ Max(t_l) & \text{if } Max(t_l) > k, Max(t_r) \\ Max(t_r) & \text{if } Max(t_r) > k, Max(t_l). \end{cases}$$