Assignment 3

Yu-Chieh Kuo B07611039[†]

[†]Department of Information Management, National Taiwan University

Problem 1

Given the theorem $(f(n))^c = o(a^{f(n)})$ for all $c > 0$, $a > 1$ and monotonically increasing $f(n)$, let $f(n) = \log_2 n$, $c = a$, and $a = 2^b$, we obtain

$$\begin{aligned} & (\log_2 n)^a = o\left((2^b)^{\log_2 n}\right) \\ \iff & (\log_2 n)^a = o\left(n^{b \log_2 2}\right) \\ \iff & (\log_2 n)^a = o\left(n^b\right). \end{aligned}$$

Problem 2

2.(a)

Given $f(n) = \frac{n^2}{\log n}$ and $g(n) = n(\log n)^2$, we claim $f(n) = \Omega(g(n))$ by testing $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$:

$$\begin{aligned} & \lim_{n\to\infty} \frac{g(n)}{f(n)} \\ = & \lim_{n\to\infty} \frac{(\log n)^3}{n} \\ = & \lim_{n\to\infty} \frac{3(\log n)^2}{n} && \text{(By L' Hopital's Rule)} \\ = & \lim_{n\to\infty} \frac{6\log n}{n} && \text{(By L' Hopital's Rule)} \\ = & \lim_{n\to\infty} \frac{6}{n} && = 0. \end{aligned}$$

Since $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0 \iff g(n) = o(f(n))$ implies $f(n) = \Omega(g(n))$, and $g(n) = o(f(n))$ implies $f(n) \neq O(g(n))$, we prove the claim.

2.(b)

Given $f(n) = n^3 2^n$ and $g(n) = 3^n$, we claim $f(n) = O(g(n))$ by testing $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$:

$$\begin{aligned}
&\lim_{n\to\infty} \frac{f(n)}{g(n)} \\
= &\lim_{n\to\infty} \frac{n^3 2^n}{3^n} \\
= &\lim_{n\to\infty} \frac{n^3}{(\frac{3}{2})^n} \\
= &\lim_{n\to\infty} \frac{3n^2}{(\frac{3}{2})^n(\ln\frac{3}{2})} \quad \text{(By L' Hopital's Rule)} \\
= &\lim_{n\to\infty} \frac{6n}{(\frac{3}{2})^n(\ln\frac{3}{2})^2} \quad \text{(By L' Hopital's Rule)} \\
= &\lim_{n\to\infty} \frac{6}{(\frac{3}{2})^n(\ln\frac{3}{2})^3} \quad = 0.
\end{aligned}$$

Since $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \iff f(n) = o(g(n))$ implies $f(n) = O(g(n))$ and $f(n) \neq \Omega(g(n))$, we prove the claim.

Problem 3

Given $f(n) = O(g(n))$, the definition of $O()$ introduces that there exists a constant c and N s.t. $\forall\, n \geq N,\ f(n) \leq cg(n)$. Taking the natural log to both sides of $f(n) \leq cg(n)$ implies

$$\begin{aligned}
&\ln(f(n)) \leq \ln(c) + \ln(g(n)) && \forall\, n \geq N \\
\iff &\frac{\ln(f(n))}{\ln(g(n))} \leq \frac{\ln(c) + \ln(g(n))}{\ln(g(n))} && \forall\, n \geq N \\
\iff &\frac{\ln(f(n))}{\ln(g(n))} \leq \frac{\ln(c)}{\ln(g(n))} + 1 && \forall\, n \geq N \\
\iff &\ln(f(n)) \leq \left(\frac{\ln(c)}{\ln(g(n))} + 1\right) \cdot \ln(g(n)) && \forall\, n \geq N \\
\iff &\ln(f(n)) \leq c' \ln(g(n)) && \text{Set } c' = \frac{\ln(c)}{\ln(g(N))} + 1 \\
\implies &\ln(f(n)) = O(\ln(g(n))).
\end{aligned}$$

Note that the operations above is feasible since $\ln(\cdot)$ is a strictly increasing function.

To verify $2^{f(n)} = O\left(2^{g(n)}\right)$, setting $f(n) = 2\log_2 n$ and $g(n) = \log_2 n$ satisfying $f(n) = O(g(n))$ implies

$$2^{2\log_2 n} = O\left(2^{\log_2 n}\right) \iff n^2 = O(n),$$

which is a contradiction. Therefore, $2^{f(n)} = O\left(2^{g(n)}\right)$ is a false statement.

Problem 4

The inequalities

$$T(n) \leq cn \quad \text{and} \quad 2T\left(\frac{n}{2}\right) + 1 \leq cn$$

implies

$$T\left(\frac{n}{2}\right) \leq c \cdot \frac{n}{2} \quad \text{and} \quad 2T\left(\frac{n}{2}\right) + 1 \leq c \cdot \frac{n}{2} + 1;$$

however, the inequalities above cannot imply $cn \geq 2c\left(\frac{n}{2}\right) + 1$, which is an incorrect statement.

Next, as the recurrence relation satisfies the form of $T(n) = aT\left(\frac{n}{b}\right) + O(n^k)$, we obtain $T(n) = O(n^{\log_2 2}) = O(n)$ by applying the master theorem.

Problem 5

We use the generating function $G(z) = \sum_{n=1}^{\infty} a_n z^{n-1}$ and observe that the recurrence relation can be represented as the form $T_n = T_{n-1} + 2T_{n-2}$. Additionally, the recurrence under such scenario is

$$G(z) = T_1 + T_2 z + T_3 z^2 + \cdots .$$

Multiplying both sides of the recurrence by $-z$ and $-2z^2$ and getting summation gives the equation

$$(1 - z - 2z^2)G(z) = 1 + 2z \quad \Longleftrightarrow \quad G(z) = \frac{1+2z}{(1+z)(1-2z)} = \frac{4}{3} \cdot \frac{1}{1-2z} - \frac{1}{3} \cdot \frac{1}{1+z},$$

which implies the recurrence relation $T(n) = 2^n + (-1)^n$.