Homework 2

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PART I

Problems

The *characteristic polynomial* $\chi(\lambda)$ of the 3×3 matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

is given by the formula

$$\chi(\lambda) = \begin{vmatrix} \lambda - a & -b & -c \\ -d & \lambda - e & -f \\ -g & -h & \lambda - i \end{vmatrix}.$$

Problem 1

To enable the following discussions, we shall assume that $a \geq 0$ and that $b \geq \theta \geq 0$. Such assumptions stem from the idea that the demand shall be nonnegative when the price $p = 0$ ($a \geq 0$), that the demand shall decrease in price ($b \geq 0$), that the two retailers shall be competing in price since the products are identical ($\theta \geq 0$), and that the total demand shall also decrease in price ($q_1 + q_2 = 2a - (b - \theta)(p_1 + p_2)$ decrease in $p_1 + p_2 \implies \theta \leq b$).

For simplicity, we also assume that c is small enough so that it is always profitable to produce products.

1.(a)

The optimization problem for retailer i is

$$\begin{array}{lll} \max\limits_{q_1, q_2, p_1, p_2} & \beta p_1 + (1-\beta) p_2 - \frac{1}{2} c q_1^2 - \frac{1}{2} c q_2^2 & \\ \text{s.t.} & \theta_1 q_1 - p_1 \geq 0 & IR-1 \\ & \theta_2 q_2 - p_2 \geq \theta_2 q_1 - p_1 & LDIC \\ & q_2 \geq q_1 & monotonicity\ . \end{array}$$

As the objective function has second derivative $u_r'' = -b < 0$, FOC is sufficient. The FOC for retailer i gives $a - bp_i^* + \theta p_{3-i}^* - bp_i^* + bw = 0$, and by symmetry we have $p_1^*(w) = p_2^*(w) = p^*(w) = \dfrac{a + bw}{2b - \theta}$. The corresponding market demand (and thus the order quantity of both retailers) is

$$q_1^*(w) = q_2^*(w) = a - (b - \theta)p^*(w) = \frac{b(a - bw + \theta w)}{2b - \theta}\ .$$

1.(b)

The optimization problem for the manufacturer is

$$\max_{w \geq c} \quad (w-c)\left(\frac{2b(a-bw+\theta w)}{2b-\theta}\right),$$

and the objective function has second derivative $u_m'' = -\dfrac{2b(b-\theta)}{2b-\theta} < 0$, thus FOC is sufficient.

The FOC gives

$$\frac{2b(a-bw^*+\theta w^*)}{2b-\theta} - \frac{2b(b-\theta)(w^*-c)}{2b-\theta} = 0$$

$$\iff \qquad w^* = \frac{a+bc-\theta c}{2(b-\theta)}.$$

The corresponding equilibrium sales quantity is $Q = q_1 + q_2 = \dfrac{2b(a-bw^*+\theta w^*)}{2b-\theta} = \dfrac{b(a-bc+\theta c)}{2b-\theta}$, and the equilibrium retail price is $p_1^* = p_2^* = p^* = p^*(w^*) = \dfrac{a+bw^*}{2b-\theta} = \dfrac{3ab+b^2c-2\theta a-b\theta c}{2(b-\theta)(2b-\theta)}$.

1.(c)

Observe that

$$\begin{aligned}\frac{\partial p^*}{\partial c} &= \frac{b^2-b\theta}{2(b-\theta)(2b-\theta)} \\ &= \frac{b}{2(2b-\theta)}.\end{aligned}$$

It can be seen that $\dfrac{\partial p^*}{\partial c} \geq 0$, i.e. the equilibrium retail price increases in the production cost, which is quite intuitive. We shall discuss 2 extreme cases to gain deeper insights.

First consider the case where $\theta = 0$, i.e. the two retailers both monopolize the market. In such case, $\dfrac{\partial p^*}{\partial c} = \dfrac{1}{4}$.

Next consider the case where $\theta = b$, i.e. the two retailers are competing intensely. In such case, $\dfrac{\partial p^*}{\partial c} = \dfrac{1}{2}$.

It can be seen that the effect of c is more significant when the competition is more intense, and such difference does not occur in w, as $\dfrac{\partial w^*}{\partial c} = \dfrac{1}{2}$ is independent of θ and b. Such difference comes from the fact that when facing a monopolized market, the retailer have more power on setting the price, and is not affected by the change in w, and thus in c, that much. When facing a highly competitive market, however, the retailer must set a price closer to w, and thus an increase in c affects the retail price more significantly.

1.(d)

Since the retailers wrongly believe that order will be fulfilled, the optimal price and order quantity would still be $p^*(w) = \dfrac{a+bw}{2b-\theta}$ and $q^*(w) = \dfrac{b(a-bw+\theta w)}{2b-\theta}$ respectively.

The manufacturer now face the following optimization problem:

$$\max_{w \geq c} \quad (w-c)\cdot \min\left(\frac{2b(a-bw+\theta w)}{2b-\theta}, K\right).$$

Note that $\dfrac{2b(a-bw+\theta w)}{2b-\theta} \leq K \iff w \geq \dfrac{2ab-K(2b-\theta)}{2b(b-\theta)}$, thus we can split the optimization problem into 2 cases.

First consider the case where w is such that the ordered quantity $Q \leq K$, i.e. all demands are met. In such case, the optimization problem is

$$\begin{aligned}\max_{w \geq c} \quad & (w-c)\cdot\frac{2b(a-bw+\theta w)}{2b-\theta} \\ \text{s.t.} \quad & w \geq \frac{2ab-K(2b-\theta)}{2b(b-\theta)},\end{aligned}$$

and the derivation is the same as that done in problem 1.(b). The optimal wholesale price is $w^* = \dfrac{a+bc-\theta c}{2(b-\theta)}$ if $K \geq \dfrac{b(a-bc+\theta c)}{2b-\theta}$. The FOC is infeasible if $K < \dfrac{b(a-bc+\theta c)}{2b-\theta}$, and in such case, the optimal wholesale price would occur when the constraint is binding, i.e. $w^* = \dfrac{2ab-K(2b-\theta)}{2b(b-\theta)}$.

The corresponding profit for the manufacturer is

$$
\begin{aligned}
u_m &= \begin{cases} \left(\dfrac{a+bc-\theta c}{2(b-\theta)} - c\right) \cdot \dfrac{b(a-bc+\theta c)}{2b-\theta} & K \geq \dfrac{b(a-bc+\theta c)}{2b-\theta} \\ \left(\dfrac{2ab-K(2b-\theta)}{2b(b-\theta)} - c\right) \cdot K & K < \dfrac{b(a-bc+\theta c)}{2b-\theta} \end{cases} \\
&= \begin{cases} \dfrac{b(a-bc+\theta c)^2}{2(b-\theta)(2b-\theta)} & K \geq \dfrac{b(a-bc+\theta c)}{2b-\theta} \\ \dfrac{K(2b(a-bc+\theta c)-K(2b-\theta))}{2b(b-\theta)} & K < \dfrac{b(a-bc+\theta c)}{2b-\theta} \end{cases}.
\end{aligned}
$$

Next consider the case where w is such that the ordered quantity $Q > K$, i.e. some demands are not met. In such case, the optimization problem is

$$
\begin{aligned}
\max_{w \geq c} \quad & (w-c) \cdot K \\
\text{s.t.} \quad & w < \frac{2ab-K(2b-\theta)}{2b(b-\theta)}.
\end{aligned}
$$

As the objective function is increasing in w, the optimal wholesale price should be as large as possible until the constraint is binding, which then falls back to the case where $w^* = \dfrac{2ab-K(2b-\theta)}{2b(b-\theta)}$. Thus, choosing any w s.t. $Q > K$ is dominated and should never be adopted.

We conclude that the optimal wholesale price should be

$$
w^* = \begin{cases} \dfrac{a+bc-\theta c}{2(b-\theta)} & K \geq \dfrac{b(a-bc+\theta c)}{2b-\theta} \\ \dfrac{2ab-K(2b-\theta)}{2b(b-\theta)} & K < \dfrac{b(a-bc+\theta c)}{2b-\theta} \end{cases},
$$

where for $K \geq \dfrac{b(a-bc+\theta c)}{2b-\theta}$, the capacity does not affect the original optimal price and thus the result is exactly the same as that derived in 1.(b), and for $K < \dfrac{b(a-bc+\theta c)}{2b-\theta}$, the wholesaler choose the highest w so that the order quantity is exactly the capacity.

To see how K affects the equilibrium outcome, we first derive the equilibrium retail price for the capacitated case:

$$
\begin{aligned}
p^* &= p^*(w^*) \\
&= \frac{a+bw^*}{2b-\theta} \\
&= \frac{a+b \cdot \frac{2ab-K(2b-\theta)}{2b(b-\theta)}}{2b-\theta} \\
&= \frac{2a-K}{2(b-\theta)}.
\end{aligned}
$$

Observe that $\dfrac{\partial p^*}{\partial K} = -\dfrac{1}{b-\theta} \leq 0$, i.e. the equilibrium retail price decreases in K, which in fact comes from the decrease in wholesale price chosen by the manufacturer in order to sell the extra products.

Problem 2

2.(a)

The retailer's optimization problem is

$$\begin{aligned} & \max_{q\geq 0} \quad \mathbb{E}[r\cdot \min(D,q)] - s\cdot \Pr[D>q] - wq \\ \iff & \max_{q\geq 0} \quad r\cdot\left(\int_0^q xf(x)\,dx + \int_q^\infty qf(x)\,dx\right) - s\cdot(1-F(q)) - wq \\ \iff & \max_{q\geq 0} \quad r\cdot\left(\int_0^q xf(x)\,dx + q(1-F(q))\right) - s(1-F(q)) - wq \\ \iff & \max_{q\geq 0} \quad r\cdot\int_0^q xf(x)\,dx + (rq-s)(1-F(q)) - wq\,. \end{aligned}$$

2.(b)

First observe that choosing any $q>b$ is always dominated by choosing $q=b$, and is thus never optimal.

First suppose that $q\in[a,b]$, and denote the optimal order quantity here by $q^\dagger$. The first derivative of the retailer's utility is

$$\begin{aligned} u_r' &= rqf(q) + r(1-F(q)) - (rq-s)f(q) - w \\ &= r(1-F(q)) + sf(q) - w \\ &= \frac{r(b-q)+s}{b-a} - w\,, \end{aligned}$$

and the second derivative is $u_r'' = -\dfrac{r}{b-a} < 0$, thus FOC is sufficient.

Solving the FOC gives $\dfrac{r(b-q^\dagger)+s}{b-a} - w = 0 \iff q^\dagger(w) = \dfrac{br-(b-a)w+s}{r}$, and this is feasible only when $a \leq \dfrac{br-(b-a)w+s}{r} \leq b \iff \dfrac{s}{b-a} \leq w \leq r + \dfrac{s}{b-a}$. When FOC is infeasible, the optimal solution would be on the boundaries.

We conclude that under the assumption that $q\in[a,b]$, the optimal order quantity is

$$q^\dagger(w) = \begin{cases} b & w \leq \dfrac{s}{b-a} \\ \dfrac{br-(b-a)w+s}{r} & \dfrac{s}{b-a} < w \leq \dfrac{(b-a)r+s}{b-a} \\ a & w > r + \dfrac{s}{b-a}\,. \end{cases}$$

Next consider the case where $q\in[0,a]$, and denote the optimal order quantity by $q^\ddagger$. The optimization problem becomes

$$\max_{0\leq q\leq a} \quad q(r-w) - s\,.$$

Clearly, the objective function is increasing in q if $r>w$, decreasing in q if $r<w$, and independent of q if $r=w$.

The optimal order quantity is then

$$q^\ddagger(w) = \begin{cases} a & w \leq r \\ 0 & w > r\,. \end{cases}$$

To combine the results derived above, observe that as the c.d.f. $F(\cdot)$ is continuous, the utility function is continuous as well. It thus suffices to discuss the first derivative (i.e. gradient) at the boundary $q=a$. We next split the case depending on the relation between r and $\dfrac{s}{b-a}$.

First consider the case where $r \leq \dfrac{s}{b-a}$. The sign of the first derivative near $q=a$ is summarized as follows.

	$w\leq r$	$r<w\leq \frac{s}{b-a}$	$\frac{s}{b-a} < w \leq r+\frac{s}{b-a}$	$r+\frac{s}{b-a} < w$
$q<a$	$+$	$-$	$-$	$-$
$q\geq a$	$+$	$+$	$+$	$-$

In such case, we have $q^* = q^\dagger = b$ when $w \le r$, and $q^* = q^\ddagger = 0$ when $w > r + \dfrac{s}{b-a}$.

For $r < w \le \dfrac{s}{b-a}$, the payoff of choosing $q = q^\ddagger = 0$ is $-s$, and that of choosing $q = q^\dagger = b$ is $\dfrac{r(b+a)}{2} - bw$. The critical $w = w_{\text{cutoff}}$ that makes choosing $q = q^\dagger$ and $q = q^\ddagger$ indifferent satisfies

$$\begin{aligned} -s &= \frac{r(b+a)}{2} - bw_{\text{cutoff}} \\ \iff \quad w_{\text{cutoff}} &= \frac{r(b+a)+2s}{2b}. \end{aligned}$$

It can be seen that $r < w_{\text{cutoff}} \le \dfrac{s}{b-a}$ always hold. Therefore, when $w \le w_{\text{cutoff}}$, the optimal order quantity is $q^* = q^\dagger = b$ and when $w > w_{\text{cutoff}}$, the optimal order quantity is $q^* = q^\ddagger = 0$. Note that as u_r must decrease in w, we may conclude that

$$q^*(w) = \begin{cases} b & w \le \dfrac{r(b+a)+2s}{2b} \\ 0 & \text{otherwise.} \end{cases}$$

Next consider the case where $r > \dfrac{s}{b-a}$. The sign of the first derivative near $q = a$ is summarized as follows.

	$w \le \frac{s}{b-a}$	$\frac{s}{b-a} < w \le r$	$r < w \le r + \frac{s}{b-a}$	$r + \frac{s}{b-a} < w$
$q < a$	$+$	$+$	$-$	$-$
$q \ge a$	$+$	$+$	$+$	$-$

Thus we have $q^* = q^\dagger$ for $w \le r$ and $q^* = q^\ddagger$ for $w > r + \dfrac{s}{b-a}$. For $r < w \le r + \dfrac{s}{b-a}$, the utility of adopting $q^\dagger$ is

$$\begin{aligned} u_r^\dagger(w) &= r \cdot \int_0^{q^\dagger} x f(x)\,dx + (q^\dagger r - s)(1 - F(q^\dagger)) - wq^\dagger \\ &= r \cdot \frac{{q^\dagger}^2 - a^2}{2(b-a)} + (q^\dagger r - s) \cdot \frac{b - q^\dagger}{b-a} - wq^\dagger \\ &= \frac{(br - (b-a)w + s)^2 - a^2 r^2}{2(b-a)r} + (br - (b-a)w) \cdot \frac{(b-a)w - s}{(b-a)r} - w \cdot \frac{br - (b-a)w + s}{r} \end{aligned}$$

Solving $u_r^\dagger = -s$ gives the cutoff wholesale price

$$w_{\text{cutoff}} = \frac{s + br \pm \sqrt{ar(ar+2s)}}{b-a},$$

where the plus term is not in the range $w \le r + \dfrac{s}{b-a}$. Using again the fact that $u_r^\dagger$ decreases in w, we conclude that

$$q^*(w) = \begin{cases} b & w \le \dfrac{s}{b-a} \\ \dfrac{br - (b-a)w + s}{r} & \dfrac{s}{b-a} < w \le \dfrac{s + br - \sqrt{ar(ar+2s)}}{b-a} \\ 0 & \text{otherwise.} \end{cases}$$

Since whether $q^* = 0$ or not depends on the choice of w made the manufacturer, no assumption is made here.

2.(c)

We first discuss the case where $r \le \dfrac{s}{b-a}$. In such case, we have

$$q^*(w) = \begin{cases} b & w \le \dfrac{r(b+a)+2s}{2b} \\ 0 & \text{otherwise,} \end{cases}$$

and the manufacturer's best policy is apparently $w^* = \frac{r(b+a)+2s}{2b}$ if $\frac{r(b+a)+2s}{2b} \geq c$, which always holds under the assumption that $r \geq c$.

Next consider the case where $r > \frac{s}{b-a}$. In such case, we have

$$q^*(w) = \begin{cases} b & w \leq \frac{s}{b-a} \\ \frac{br-(b-a)w+s}{r} & \frac{s}{b-a} < w \leq \frac{s+br-\sqrt{ar(ar+2s)}}{b-a} \\ 0 & \text{otherwise.} \end{cases}$$

Observe that optimization problem the manufacturer faces is

$$\max_w \quad \frac{br-(b-a)w+s}{r} \cdot (w-c)$$

with the demand saturates at b for $w \leq \frac{s}{b-a}$, drops to 0 for $w > \frac{s+br-\sqrt{ar(ar+2s)}}{b-a}$. It thus suffices to solve for the optimal w between $\frac{s}{b-a}$ and $\frac{s+br-\sqrt{ar(ar+2s)}}{b-a}$, which is in fact a CP.

FOC gives $w^* = \frac{br+s+(b-a)c}{2(b-a)}$. While $w^* > \frac{s}{b-a}$ always holds, $w^* \leq \frac{s+br-\sqrt{ar(ar+2s)}}{b-a}$ is violated when $s > 3ar-(b-a)(r-c)-2\sqrt{ar(3ar-2(b-a)(r-c))}$, and in such case $w^* = \frac{s+br-\sqrt{ar(ar+2s)}}{b-a}$.

The optimal wholesale price w^* is summarized as

$$w^* = \begin{cases} \frac{br+s+(b-a)c}{2(b-a)} & s \leq 3ar-(b-a)(r-c)-2\sqrt{ar(3ar-2(b-a(r-c)))} \\ \frac{s+br-\sqrt{ar(ar+2s)}}{b-a} & 3ar-(b-a)(r-c)-2\sqrt{ar(3ar-2(b-a)(r-c))} < s \leq (b-a)r \\ \frac{r(b+a)+2s}{2b} & s > (b-a)r \end{cases}$$

2.(d)

The impact of s on w^* is summarized as

$$\frac{\partial w^*}{\partial s} = \begin{cases} \frac{1}{2(b-a)} & s \leq 3ar-(b-a)(r-c)-2\sqrt{ar(3ar-2(b-a(r-c)))} \\ \frac{ar+2s-\sqrt{ar(ar+2s)}}{(b-a)(ar+2s)} & 3ar-(b-a)(r-c)-2\sqrt{ar(3ar-2(b-a)(r-c))} < s \leq (b-a)r \\ \frac{1}{b} & s > (b-a)r\,. \end{cases}$$

For fairly small s, the wholesaler's behavior is still similar to the case without s, and an increase in s leads to an increase in the order quantity, and thus an increase in the equilibrium wholesale price w^*.

For extremely large s, on the other hand, the wholesaler chooses the maximum w^* such that the retailer is indifferent between ordering b (meeting all demands) and 0 (giving up). Thus the effect of s is inverse proportional to b.

For s in between, the wholesaler can no longer choose the w^* that satisfies the FOC, as this would cause the retailer to prefer ordering nothing at all. Thus the retailer would again choose the maximum w^* such that the retailer is indifferent between ordering or not.

In all three cases, the equilibrium wholesale price increases with s, as the retailer wants to meet the demand more eagerly and is thus willing to take a higher price.

2.(e)

When $s = 0$, the equilibrium wholesale price is $w^* = \frac{br+(b-a)c}{2(b-a)}$, and the corresponding order quantity is $q^* = \frac{br-(b-a)w}{r}$. This is indeed the case derived in class, where q^* is such that $1-F(q^*) = \frac{w}{r}$.

As for the case where $s \to \infty$, the equilibrium wholesale price is $w^* = \frac{r(b-a)+2s}{2b}$, and the corresponding order quantity is $q^* = b$. The wholesaler's earning is approximately $\frac{s}{b}$ for large enough s, and goes to infinity as s grows to infinity.

PART II

Case Study

Section 1: Research Topic and Assumptions

We believe that the major benefit of using mobile payment is that consumers have heterogeneous preference over payment options, and thus providing different payment options would larger the demand. Moreover, we assume such difference is *horizontal*, i.e. the whole market does not favor one payment method over another in aggregate.

We assume the retailer has time preference across periods, and that once the platform is set up, only a fixed maintenance cost is needed for each period. Although the one-time setup cost may be one of the major concerns of setting up a new payment platform, it can be combined with the maintenance cost using discount factor in each period and focus on one period only. By doing so, the model can be focused on only 1 period, say 1 year.

On the other hand, we assume the platform already has a working service, and the setup cost is sunk and thus not in consideration.

To further simplify the derivation, we assume the unit production cost is 0, and that the retailer cannot set different price for different payment methods. We also assume that no uncertainty or information asymmetry exists, and that the platform has all bargaining power over the retailer.

Section 2: The Model

The model considers a mobile payment service provider (called the platform in the sequel), a retailer and a group of heterogenous consumers. The retailer is selling one product only. Each consumer is endowed with $x \in R$, and they lie uniformly with density 1. Each of them obtain a utility of $V - p$ if getting the product at price p. However, consumer x must incur a mental cost of $|x|$ if he pays by cash due to inconvenience.

The platform is endowed with a mobile payment platform characterized by $X_P > 0$, where consumer x incurs a mental cost of $|x - X_P|$ if he pays using the service.

If the retailer joins the platform, he has to pay a fixed fee f to the platform as well as ϕp for each product paid by mobile payment, and the platform would incur a maintenance cost C_P.

On the other hand, if the retailer chooses to set up a new platform, he would incur a setup cost $C_{S,R}$ and maintenance cost $C_{M,R}$. However, as he does not need to set it up again the next period, he views it as if the total cost is $C_R = C_{M,R} + (1-\delta)C_{S,R}$ for each period, where $\delta \in [0, 1]$ is the discount factor of the retailer.

The new platform is characterized by $X_R > 0$, where consumer x incurs a mental cost of $|x - X_R|$ if he pays using the service.

The game proceeds as follows. First, the platform offers a contract (f, ϕ) to the retailer. Given the contract, the retailer decides whether to accept the contract and join the platform, to set up a new platform, or keep on accepting cash only. The retailer then announce its retail price p, and each consumer buys a product with the payment method generating the highest nonnegative utility for him, or nothing if all payment methods generates a negative utility.

Section 3: Model Solving

We shall begin with the subgame where no mobile payment exists. In such case, consumer x will buy the product if and only if $V - p - |x| \geq 0 \iff p - V \leq x \leq V - p$, i.e. the demand is $2(V - p)$ whenever $p \leq V$ and 0 otherwise. The optimization problem for the retailer is

$$\max_{p \leq V} \quad 2p(V-p)\,,$$

and the optimal price is $p_N^* = \dfrac{V}{2}$, with payoff $\pi_N^* = \dfrac{V^2}{2}$.

Next consider the case where the retailer joins the platform. In such case, consumer x will buy the product if and only if $\max(V - p - |x|, V - p - |x - X_P|) \geq 0$, which is true if either $p - V \leq x \leq V - p$ or $X_P + p - V \leq x \leq X_P + V - p$.

If the retailer chooses $p \geq V - \dfrac{X_P}{2}$, then the two segments would be disjoint, thus the demand for each payment method is $2(V - p)$ whenever $p \leq V$ and 0 otherwise. The optimization problem for the retailer is

$$\max_{V - \frac{X_P}{2} \leq p \leq V} \quad 2(2 - \phi)p(V - p)\,,$$

and the FOC gives $p_P^\dagger = \dfrac{V}{2}$, which is feasible only when $X_P \geq V$. If $X_P < V$, the optimal price should be the boundary, which is $p_P^\dagger = V - \dfrac{X_P}{2}$.

We conclude that the optimal price in range $\left[V - \dfrac{X_P}{2}, V\right]$ is

$$p_P^\dagger = \begin{cases} \dfrac{V}{2} & X_P \geq V \\ V - \dfrac{X_P}{2} & \text{otherwise,} \end{cases}$$

with payoff

$$\pi_P^\dagger = \begin{cases} \dfrac{(2-\phi)V^2}{2} & X_P \geq V \\ (2-\phi)X_P\left(V - \dfrac{X_P}{2}\right) & \text{otherwise.} \end{cases}$$

If the retailer chooses $p \leq V - \dfrac{X_P}{2}$ instead, then the two payment method would equally share the region between 0 and X_P. The demand for each payment method would thus be $V - p + \dfrac{X_P}{2}$. The optimization problem for the retailer becomes

$$\max_{p \leq V - \frac{X_P}{2}} \quad (2 - \phi)p\left(V - p + \frac{X_P}{2}\right),$$

and the FOC gives $p_P^\ddagger = \dfrac{V}{2} + \dfrac{X_P}{4}$, which is feasible only when $X_P \leq \dfrac{2V}{3}$. If $X_P > \dfrac{2V}{3}$, the optimal price should be the boundary, which is $p_P^\ddagger = V - \dfrac{X_P}{2}$.

We conclude that the optimal price in range $\left[0, V - \dfrac{X_P}{2}\right]$ is

$$p_P^\ddagger = \begin{cases} \dfrac{V}{2} + \dfrac{X_P}{4} & X_P \leq \dfrac{2V}{3} \\ V - \dfrac{X_P}{2} & \text{otherwise,} \end{cases}$$

with payoff

$$\pi_P^\ddagger = \begin{cases} \dfrac{(2-\phi)(2V + X_P)^2}{16} & X_P \leq \dfrac{2V}{3} \\ (2-\phi)X_P\left(V - \dfrac{X_P}{2}\right) & \text{otherwise.} \end{cases}$$

Since the feasible region of FOC are disjoint, the two results can be combined directly, giving the optimal price

$$p_P^* = \begin{cases} \dfrac{V}{2} + \dfrac{X_P}{4} & X_P \leq \dfrac{2V}{3} \\ V - \dfrac{X_P}{2} & \dfrac{2V}{3} < X_P < V \\ \dfrac{V}{2} & X_P \geq V \end{cases}$$

and the corresponding payoff

$$\pi_P^* = \begin{cases} \dfrac{(2-\phi)(2V+X_P)^2}{16} - f & X_P \le \dfrac{2V}{3} \\ (2-\phi)X_P\left(V - \dfrac{X_P}{2}\right) - f & \dfrac{2V}{3} < X_P < V \\ \dfrac{(2-\phi)V^2}{2} - f & X_P \ge V\,. \end{cases}$$

The subgame where the retailer sets up his own mobile service is essentially substituting the above X_P with X_R, ϕ with 0 and f with C_R. Thus the optimal price is

$$p_R^* = \begin{cases} \dfrac{V}{2} + \dfrac{X_R}{4} & X_R \le \dfrac{2V}{3} \\ V - \dfrac{X_R}{2} & \dfrac{2V}{3} < X_R < V \\ \dfrac{V}{2} & X_R \ge V\,, \end{cases}$$

and the corresponding payoff is

$$\pi_R^* = \begin{cases} \dfrac{(2V+X_R)^2}{8} - C_R & X_R \le \dfrac{2V}{3} \\ X_R(2V - X_R) - C_R & \dfrac{2V}{3} < X_R < V \\ V^2 - C_R & X_R \ge V\,. \end{cases}$$

It can be seen that the revenue sharing ratio ϕ is *neutral* in our model, i.e. it has no effect on the retailer's choice of V. This result is discussed in Section 4, and we shall first make use of this property and assume WLOG that the platform chooses $\phi = 0$. Under such assumption, the platform's proposal must satisfies $f \ge C_P$ so that the contract generates positive utility for it.

We first analyze when introducing mobile payment is efficient. For a platform characterized by (X, C), it is efficient if and only if the generated profit minus the cost C is greater than $\dfrac{V^2}{2}$. To ease notation, we define the *efficiency* of a platform (X, C) as

$$\eta \coloneqq \begin{cases} \dfrac{X(4V+X)}{8} - C & X \le \dfrac{2V}{3} \\ \dfrac{V^2 - 2(V-X)^2}{2} - C & \dfrac{2V}{3} < X < V \\ \dfrac{V^2}{2} - C & X \ge V\,. \end{cases}$$

If both η_P and η_R are negative, then the retailer will keep accepting cash only. Otherwise, the platform with higher η will be introduced, and in case that η_P is higher, the contract will be designed such that the retailer is indifferent between accepting or not[1].

Section 4: Results and Discussions

We first focus on the difference between using mobile payment or not by looking at p_R^* and π_R^*. First, it can be seen that $p_R^* \ge p_N^* = \dfrac{V}{2}$ always holds. Moreover, as X_R increases, the equilibrium price first increases to $\dfrac{2V}{3}$, and then drops back to $\dfrac{V}{2}$ and stays fixed.

In the first region, the two payment methods are not quite separated, thus the consumers with $0 \le x \le X_R$ are always willing to buy the product. In such case, the price can be higher, as the demand is not as sensitive to price as before.

[1] If $\eta_R \ge 0$, the equilibrium profit for the retailer is that setting up his own service. Otherwise, it would be $\dfrac{V^2}{2}$.

In the second region, it is optimal to choose a price such that the consumer with $x = \frac{X_R}{2}$ is indifferent between buying or not, as the gradient of the demand has a jump here.

In the third region, the two payment methods completely separated when the price is optimal, and thus it is optimal to choose the same optimal price as in the original case.

One assumption worth noting is that the price for both payment must be the same. This lead to the result that the revenue sharing ratio ϕ being *neutral*, in a sense that it doesn't affect the pricing strategy just as f. We believe that this is quite common based on personal experience, and it may be justify by either assuming consumer's utility about fairness, or simply assume it is part of the contract.

If the assumption is taken away from the model, the pricing strategy would depend on ϕ, as it may be beneficial to the retailer to encourage more consumers to pay by cash, thus resulting in an inefficiently high price for mobile payment. Nonetheless, the platform would then optimally chooses $\phi^* = 0$ and charge by f only, leading to the same equilibrium outcome.

Why revenue sharing is widely adopted cannot be explained by this model. We believe the reasons may include asymmetric belief on the demand, the time preference on revenue sharing over the one-time fee f, or risk-aversion under random demand. All reasons mentioned are mere hypothesis, and justifying them requires further modeling and analysis which are not included in this research.

Finally, although the model seems to be simplifying the choice between joining a platform or setting up a new one as a mere efficiency issue, some other issues such as liquidity may still be more or less captured by the retailer's discount factor δ, and thus affect the cost C_R and reflect on the efficiency factor η_R.